# ON QUASI-PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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A quasi-periodic function, in contradistinction to a periodic function, has several periods [1]. Therefore, in the investigation of quasi-periodicity it is natural to consider a space of several dimensions.

In the study of solutions of ordinary differential equations the author passes from the given system of equations to a special system of partial differential equations [2-4]. This makes it possible to reduce the problem on quasi-periodic oscillations to the problem of periodic oscillations. In this manner, it is possible to characterize quasi-periodicity with the aid of boundary conditions.

In the first part of this work, this method is used for the investigation of linear systems with quasi-periodic coefficients, and the structure of the solutions of such systems is established. In the second part, nonlinear systems are considered.

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**1. Linear systems of equations.** 1. Let the equation

$$\frac{dx}{dt} = P(t)\,x, \qquad P(t) = \| p_{sk}(t) \|_1^n \tag{1.1}$$

be given. Here $P(t)$ is a quasi-periodic matrix with a frequency basis $\beta(\beta_1, \ldots, \beta_m)$ while $x(x_1, \ldots, x_n)$ is a vector. It is known [1] that the quasi-periodic function $p_{sk}(t)$ is a diagonal matrix of the periodic function $F_{sk}(u_1, \ldots, u_m)$ with the periods $\omega_k = 2\pi/\beta_k$ in the variables $u_k$, i.e. $p_{sk}(t) = F_{sk}(t, \ldots, t)$.

Let $D$ be an operator which acts on the function $x(u_1, \ldots, u_m)$ in the following way

$$Dx = \frac{\partial x}{\partial u_1} + \frac{\partial x}{\partial u_2} + \ldots + \frac{\partial x}{\partial u_m}$$

We now construct the equation

$$Dx = F(u_1, \ldots, u_m)\, x \tag{1.2}$$

$$(F(u_1, \ldots, u_m) = \| F_{sk}(u_1, \ldots, u_m) \|_1^n, \ F(t, \ldots, t) = P(t))$$

Equation (1.2) is equivalent to equation

$$\sum_{i=1}^{m} \frac{\partial z}{\partial u_i} + \sum_{j=1}^{n} b_j \frac{\partial z}{\partial x_j} = 0 \qquad (b_j = F_{j1}x_1 + \ldots + F_{jn}x_n)$$

where $z$ is a scalar solution function. Thus the solutions of equation (1.2) will exist if there exist first integrals of the system

$$du_1 = du_2 = \ldots = du_m = \frac{dx_1}{b_1} = \ldots = \frac{dx_n}{b_n}$$

If we consider equation (1.2) along the diagonal $u_1 = u_2 = \ldots = u_m = t$ then we obtain equation (1.1). Therefore, if $x(u_1, \ldots, u_m)$ is some solution of equation (1.2), then $x(t, \ldots, t)$ is a solution of (1.1). It is also easy to see [4] that through each curve that does not lie on the $(n + 1)$-dimensional plane $u_1 = u_2 = \ldots = u_m$ in the $(m + n)$-dimensional space, there passes a unique solution of equation (1.2) which exists for all values $u_1, \ldots, u_m$, while through a curve which is a solution of equation (1.1) there passes an infinite set of solutions of equation (1.2).

2. Suppose that $F(u_1, \ldots, u_m)$ in equation (1.2) is some variable matrix (not necessarily periodic). For equation (1.2) one can construct a theory analogous to the corresponding theory for equation (1.1). Let the vectors

$$\psi_1(x_{11}, \ldots, x_{n1}), \ldots, \psi_n(x_{1n}, \ldots, x_{nn}) \qquad (x_{sk} = x_{sk}(u_1, \ldots, u_m)) \tag{1.3}$$

represent $n$ particular solutions of equation (1.2).

*Definition.* The system of solutions (1.3) will be called a fundamental system if it satisfies the following condition: let $\bar{\psi}(x_1, \ldots, x_n)$ $(x_k = x_k(u_1, \ldots, u_m))$ be a given solution of equation (1.2); then there exists differentiable functions $A_k(u_2 - u_1, \ldots, u_m - u_1)$ $(k = 1, \ldots, n)$ such that

$$\psi = A_1\psi_1 + \ldots + A_n\psi_n$$

Let $x(u_1, \ldots, u_m)$ be a matrix which consists of the functions (1.3), and let $|x(u_1, \ldots, u_m)|$ be its determinant.

*Theorem 1.1.* If $|x(u_1, \ldots, u_m|$ is different from zero for all values of $u_k$, then system (1.3) is a fundamental system.

In what follows we shall call the function $f(u_2 - u_1, \ldots, u_m - u_1)$, which depends on the differences $u_k - u_1$, the function which is constant along the diagonal. Analogously, we shall call a matrix to be constant on the diagonal if its elements are functions which are constant on the diagonal.

If the matrix $F$ in equation (1.2) is constant on the diagonal, or constant everywhere, then the matrix

$$x = \exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} F\right] (\alpha_j = \text{const})$$

will be a fundamental matrix of the solutions of equation (1.2). Hereby the structure of the solutions of equation (1.2), with a matrix $F$ that is constant along the diagonal or constant everywhere, has been completely established.

3. If one subjects equation (1.2) to the transformation

$$x = B\ (u_1, \ldots, u_m)\, Y \tag{1.4}$$

where $B$ is a nonsingular matrix, then one obtains equation

$$DY = [B^{-1}FB - B^{-1}DB]\, Y \tag{1.5}$$

Suppose that the nonsingular matrix $B$, together with the matrix $DB$, be bounded for all positive values of $u_1, \ldots, u_m$, that lie on the diagonal. A matrix $B$ which possesses this property will be called a Liapunov matrix.

If $B(u_1, \ldots, u_m)$ in the representation (1.4) is a matrix that is constant on the diagonal or constant everywhere, then equation (1.5) will take on the most simple form $DY = B^{-1}FBY$.

*Definition.* Equation (1.2) will be said to be reducible, if it can be transformed by means of the transformation (1.4) with a Liapunov matrix $B$ into an equation with a matrix that is constant on the diagonal or constant everywhere.

It is easy to obtain the theorem of Erugin [5].

*Theorem 1.2.* In order that equation (1.2) may be reducible it is necessary and sufficient that the fundamental system of the solutions of this equation be representable in the form

$$x(u_1, \ldots, u_m) = B(u_1, \ldots, u_m) \exp \left[ \frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} A \right]$$

where $B$ is a Liapunov matrix, $\alpha_k$ = const and $A$ is a matrix that is constant on the diagonal or constant everywhere.

*Theorem 1.3.* If the coefficients $F_{sk}$ in equation (1.2) are periodic function with the same real period $\omega$ in all the variables $u_k$, then equation (1.2) is reducible with the aid of a periodic matrix.

The proofs of these theorems will not be given; they can be obtained easily from known proofs for systems of ordinary equations.

If one considers equation (1.2) on the diagonal $u_k = t$, then Theorem 1.3 yields a known theorem of Liapunov on the reducibility of equation (1.1) with periodic (with a common period) coefficients.

We note that if $x_1(u_1, \ldots, u_m)$ is a fundamental matrix of solutions of equation (1.2), then the matrix

$$x(u_1, \ldots, u_m) = x_1(u_1, \ldots, u_m) B \tag{1.6}$$

where $B$ is a nonsingular matrix constant on the diagonal, is also a fundamental matrix of solutions, and all fundamental matrices of solutions of equations (1.2) are contained in formula (1.6).

4. Suppose that in equations (1.2) $F(u_1 + \omega_1, \ldots, u_m + \omega_m) = F(u_1, \ldots, u_m)$. Then, along the diagonal, equation (1.2) will yield, as was already mentioned, equation (1.1) with a quasi-periodic matrix $P(t) = F(t, \ldots, t)$.

Let $x(u_1, \ldots, u_m)$ be a fundamental matrix of solutions of equation (1.2). Then $x(u_1 + \omega_1, \ldots, u_m + \omega_m)$ will also be a solution of equation (1.2) because of the periodicity of the matrix $F$, and on the basis of what was said above, we have

$$x(u_1 + \omega_1, \ldots, u_m + \omega_m) = x(u_1, \ldots, u_m) C \tag{1.7}$$

where $C$ is a matrix constant on the diagonal or constant everywhere.

In this manner we obtain a relation which is analogous to the one which permitted Floquet [5] to determine the structure of the solutions of systems of ordinary differential equations with periodic coefficients.

*Theorem 1.4.* If equation (1.2) is such that the matrix $C$ in relation (1.7) is everywhere constant, then this equation is reducible to an equation with a matrix that is everywhere constant by means of a periodic matrix.

Indeed, let the everywhere constant matrix $B$ be such that

$$C = \exp\left[\frac{\alpha_1\omega_1 + \ldots + \alpha_m\omega_m}{\alpha_1 + \ldots + \alpha_m} B\right] \qquad (\alpha_j = \text{const})$$

Let us consider the matrix

$$K(u_1, \ldots, u_m) = \exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} B\right] x^{-1}(u_1, \ldots, u_m)$$

Then $K(u_1 + \omega_1, \ldots, u_m + \omega_m) = K(u_1, \ldots, u_m)$, i.e. the matrix $K$ is periodic. Let us set

$$Y = Kx = \exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} B\right]$$

Then $DY = BY$. We have a converse to the last theorem.

*Theorem 1.5.* If equation (1.2) is reducible to an equation with an everywhere constant matrix by means of a periodic matrix, then the matrix $C$ in relation (1.7) will be everywhere constant.

Indeed, suppose that by means of the transformation $x = BY$, where $B$ is a periodic matrix, equation (1.2) is reduced to the form $DY = AY$, where $A$ is a matrix that is everywhere constant. Then we obtain successively

$$Y = \exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} A\right], \qquad x = B\exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} A\right]$$

and, hence, everywhere the matrix

$$C = \exp\left[\frac{\alpha_1\omega_1 + \ldots + \alpha_m\omega_m}{\alpha_1 + \ldots + \alpha_m} A\right] = \text{const}$$

Applying the obtained result to equation (1.1) we obtain the next theorem.

*Theorem 1.6.* In order that equation (1.1) may be reducible, by means of a quasi-periodic matrix, it is necessary and sufficient that in relation

$$x(t + \omega_1, \ldots, t + \omega_m) = x(t, \ldots, t) M$$

the constant matrix $M$ be independent of the period $\omega_k$.

If one studies the structure of the solutions of equation (1.2) with a periodic matrix $F(u_1, \ldots, u_m)$, then one investigates at the same time the structure of the solutions of equation (1)1) with a quasi-periodic matrix $P(t) = F(t, \ldots, t)$. But the structure of the solutions of equation (1.2) is determined completely by the matrix $C$ in relation (1.7).

We shall point out one auxiliary proposition. In order that the differential function $\Phi_1(u_1, \ldots, u_m)$ satisfy the condition

$$\Phi_1(u_1+\omega_1, \ldots, u_m+\omega_m) = \Phi_1(u_1, \ldots, u_m) + p(u_2-u_1, \ldots, u_m-u_1) \quad (1.8)$$

it is necessary and sufficient that

$$D\Phi_1(u_1, \ldots, u_m) = K(u_1, \ldots, u_m) \quad (1.9)$$

where $K$ is a periodic function of the variables $u_j$ of period $\omega_j$. In particular, if $K = 0$, then $\Phi_1$ is a function that is constant on the diagonal.

Indeed, suppose that (1.9) is valid. Then

$$D[\Phi_1(u_1+\omega_1, \ldots, u_m+\omega_m) - \Phi_1(u_1, \ldots, u_m)] = 0$$

Since any differentiable function of the form $p = p(u_2 - u_1, \ldots, u_m - u_1)$ is a solution of $DZ = 0$, the sufficiency is proved.

Let us assume that (1.8) is valid. Then $D\Phi_1(u_1 + \omega_1, \ldots, u_m + \omega_m) = D\Phi_1(u_1, \ldots, u_m)$, and the necessity is also established.

Analogously, in order that $\Phi_2(u_1, \ldots, u_m)$ satisfy the condition

$$\Phi_2(u_1+\omega_1, \ldots, u_m+\omega_m) = \\ = \Phi_2(u_1, \ldots, u_m) + p(u_2-u_1, \ldots, u_m-u_1)\,\Phi_1(u_1, \ldots, u_m) \quad (1.10)$$

where $\Phi_1$ satisfies (1.8), it is necessary and sufficient that

$$D[K_1 D\Phi_2] = K_2 \quad (1.11)$$

where $K_1 \not\equiv 0$ and $K_2$ are periodic functions. From (1.11) it follows that if $\Phi_2$ is to satisfy condition (1.10) it is necessary and sufficient that $D\Phi_2 = KN_1$, where $K_1$ is periodic, while $N_1$ is a function of the type $\Phi_1$. In particular if $K_2 = 0$, then the function $N_1$ will be constant on the diagonal. Continuing in this manner, we obtain the result: in order that the function $\Phi_s(u_1, \ldots, u_m)$ may satisfy the condition

$$\Phi_s(u_1+\omega_1, \ldots, u_m+\omega_m) = \\ = \Phi_s(u_1, \ldots, u_m) + p(u_2-u_1, \ldots, u_m-u_1)\,\Phi_{s-1}(u_1, \ldots, u_m)$$

it is necessary and sufficient that

$$D\Phi_s = KN_{s-1}(u_1, \ldots, u_m)$$

where $K$ is a periodic function, and $N_{s-1}$ is a function of the type $\Phi_{s-1}$.

Let us consider the possible structures of the matrix $C$ in the relation (1.7).

1. The matrix $C$ is everywhere constant. Then it follows from Theorem 1.4 that

$$x(u_1, \ldots, u_m) = B(u_1, \ldots, u_m) \exp\left[\frac{\alpha_1 u_1 + \ldots + \alpha_m u_m}{\alpha_1 + \ldots + \alpha_m} A\right] \qquad (\alpha_j = \text{const})$$

where $B$ is periodic and $A$ is an everywhere constant matrix. For (1.1) this yields

$$x(t) = \Phi(t) \exp(tA) \tag{1.12}$$

where $\Phi(t) = B(t, \ldots, t)$ is a quasi-periodic matrix.

In this manner we obtain a result which is similar to the results of Floquet for equation (1.1) with a periodic matrix $P(t)$. This case has been investigated at length in [4].

2. The matrix $C = [\lambda_1, \ldots, \lambda_n]$, where $\lambda_k$ is a function that is constant on the diagonal. In this case we obtain the relation

$$x_{jk}(u_1 + \omega_1, \ldots, u_m + \omega_m) = \lambda_k x_{jk}(u_1, \ldots, u_m)$$

which in view of the auxiliary proposition, is satisfied by functions of the type $\Phi_{jk}(u_1, \ldots, u_m) \exp[R_k(u_1, \ldots, u_m)]$, where $\Phi_{jk}$ is periodic in the variables $u_i$ with periods $\omega_i$, while the $R_k$ are functions satisfying condition (1.9). In particular

$$R_k = \int_0^{u_1} M_k(\tau, u_2 - u_1 + \tau, \ldots, u_m - u_1 + \tau)\, d\tau$$

where the $M_k$ are periodic functions. For equation (1.1) this yields

$$x_{jk}(t) = \varphi_{jk}(t) \exp[m_k(t)] \quad \text{or} \quad x_{jk}(t) = \varphi_{jk}(t) \exp\left[\int_0^t l_k(\tau)\, d\tau\right]$$

$$(\varphi_{jk}(t) = \Phi_{jk}(t, \ldots, t), \qquad l_k(t) = M_k(t, \ldots, t)$$

where the $m_k'(t)$ are quasi-periodic functions. If

$$\int_0^t l_k(\tau)\, d\tau = at + \psi_k(t)$$

where the $\psi_k(t)$ are quasi-periodic functions, then we obtain again the form (1.2), while if the $\psi_k(t)$ are not quasi-periodic, we obtain forms of the solutions of equation (1.1) which do not coincide with the theory of Floquet. In the last case equation (1.1) is not reducible, but if the characteristic numbers (of Liapunov) for this equation and for its adjoint equation are denoted by $\alpha_k$ and $\beta_k$, respectively, then $\alpha_k + \beta_k = 0$, i.e. equation (1.1) is the correct one [5].

3. The matrix

$$C = [I_{q_1}(\lambda_1), \ldots, I_{q_p}(\lambda_p)], \qquad I_{q_k}(\lambda_k) = \left\| \begin{matrix} \lambda_k & 1 & 0 \ldots 0 \\ 0 & \lambda_k & 1 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots \lambda_k \end{matrix} \right\|$$

where the $\lambda_k$ are functions that are constant on the diagonal. Relation (1.7) now has the form

$$x(u_1 + \omega_1, \ldots, u_m + \omega_m) = x(u_1, \ldots, u_m)[I_{q_1}(\lambda_1), \ldots, I_{q_p}(\lambda_p)]$$

that is, it will consist of several groups.

For example, the group which corresponds to $\lambda_k$ will be

$$x_{sj}(u_1 + \omega_1, \ldots, u_m + \omega_m) = x_{sj-1}(u_1, \ldots, u_m) + \lambda_k x_{sj}(u_1, \ldots, u_m)$$
$$(s = 1, \ldots, n;\ j = 1, \ldots, q_k)$$

In the same way we have from the auxiliary proposition

$$x_{s1}(u_1, \ldots, u_m) = \Phi_{s1}(u_1, \ldots, u_m) \exp[R_k] \qquad (s = 1, \ldots, n)$$
$$x_{s2}(u_1, \ldots, u_m) = (\Phi_{s2} + M_{s1}\Phi_{s1}) \exp[R_k]$$
$$x_{s3}(u_1, \ldots, u_m) = (\Phi_{s3} + M_{s1}\Phi_{s2} + M_{s2}\Phi_{s1}) \exp[R_k]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{sq_k}(u_1, \ldots, u_m) = (\Phi_{sq_k} + M_{s1}\Phi_{sq_k-1} + \ldots + M_{sq_k-1}\Phi_{s1}) \exp[R_k]$$

where $\Phi_{si}(u_1, \ldots, u_m)$ is a periodic function in the variables $u_k$ with periods $\omega_k$, while the functions $M_{s1}, \ldots, M_{sq_k-1}$ satisfy the conditions

$$DM_{s1} = \alpha_1, \quad DM_{s2} = \alpha_2 N_{s1}, \ldots, DM_{sq_k-1} = \alpha_{q_k-1} N_{sq_k-2}$$

where the $\alpha_i$ are periodic functions, while the $N_{si}$ are functions of the type $M_{si}$.

The functions $R_k$ satisfy condition (1.9). For equation (1.1) this yields

$$x_{s1}(t) = \varphi_{s1}(t) \exp[r_k(t)]$$
$$x_{s2}(t) = (\varphi_{s2}(t) + h_{s1}(t)\varphi_{s1}(t)) \exp[r_k(t)]$$
$$x_{s3}(t) = (\varphi_{s3}(t) + h_{s1}(t)\varphi_{s2}(t) + h_{s2}(t)\varphi_{s1}(t)) \exp[r_k(t)]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{sq_k}(t) = (\varphi_{sq_k} + h_{s1}\varphi_{sq_k-1} + \ldots + h_{sq_k-1}\varphi_{s1}) \exp[r_k(t)]$$

where the $\varphi_{si}(t)$ and the $r_k'(t)$ are quasi-periodic functions, and the functions $h_{si}(t)$ satisfy the following conditions: $h_{si}'(t)$ is a quasi-periodic function, $h_{s2}'(t)$ is the product of a quasi-periodic function by a function whose derivative is quasi-periodic, and so on.

Equation (1.1) is again the correct one. Thus we have given a broad class of correct equations which, however, are not reducible in general.

**2. Nonlinear systems of equations.** 1. Let us consider the equation

$$dx / dt = f(t, x) + \alpha F(t, x, \alpha) \tag{2.1}$$

where $f(f_1, \ldots, f_k)$, $F(F_1, \ldots, F_n)$, and $x(x_1, \ldots, x_n)$ are vectors. The functions $f_s(t, x_1, \ldots, x_n)$ and $F_s(t, x_1, \ldots, x_n, \alpha)$ are defined for all values of $x_1, \ldots, x_n$ which lie in some region $G$ of the space of these variables. With respect to the independent variable $t$, these functions are quasi-periodic with a common frequency basis $\beta(\beta_1, \ldots, \beta_m)$. The quantity $\alpha$ is a small parameter. In accordance with the proposed method of investigation, we must consider along with equation (2.1) also equation

$$Dx = A(u_1, \ldots, u_m, x) + \alpha B(u_1, \ldots, u_m, x, \alpha) \tag{2.2}$$

$$A(t, \ldots, t, x) = f(t, x), \qquad B(t, \ldots, t, x, \alpha) = F(t, x, \alpha)$$

The functions $A_s(u_1, \ldots, u_m, x)$ and $B(u_1, \ldots, u_m, x. \alpha)$ are periodic in the variables $u_k$ with the periods $\omega_k = 2\pi/\beta_k$. In an analogous way, as above, we see that if we consider equation (2.2) on the diagonal $u_k = t$, then we obtain equation (2.1) and if $x(x_1, \ldots, x_n)$ $(x_s = x_s(u_1, \ldots, u_m))$ is some solution of equation (2.2) then it will generate on the diagonal a solution $x_s(t, \ldots, t)$ of equation (2.1).

Conversely, through every solution of equation (2.1) there pass an infinite number of solutions of equation (2.2).

Equation (2.2) is equivalent to a linear equation; therefore, through every curve that does not lie in the plane $u_1 = \ldots = u_m$ there passes a unique solution of equation (2.2). We note that if equation (2.2) has a periodic solution of period $\omega_k$ in the variables $u_k$, then equation (2.1) has a quasi-periodic solution with the frequency basis $\beta$.

Let us consider the equation

$$Dx = A(u_1, \ldots, u_m, x) \tag{2.3}$$

which we shall call the generating equation, and let us assume that this equation has a periodic solution

$$\varphi(\varphi_1, \ldots, \varphi_n) \qquad (\varphi_s = \varphi_s(u_1, \ldots, u_m)) \tag{2.4}$$

of period $\omega_k$ in the variables $u_k$, and that this solution satisfies the initial conditions

$$\varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1\right) = \psi_s(u_1) \quad \text{if } \frac{u_1}{\omega_1} = \frac{u_2}{\omega_2} = \ldots = \frac{u_m}{\omega_m}$$

where $\psi_s(u_1)$ is a periodic function of period $\omega_1$.

We shall try to find conditions under which system (2.2) admits a periodic solution which will become the generating solution (2.4) when $\alpha = 0$.

Let us denote by

$$x_s(u_1, \ldots, u_m, \eta_1(u_1, \alpha), \ldots, \eta_n(u_1, \alpha), \alpha)$$

the solution of equation (2.2) with the initial conditions

$$x_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \eta_1(u_1, \alpha), \ldots, \eta_n(u_1, \alpha), \alpha\right) = \eta_s(u_1, \alpha) \quad (2.5)$$

It is obvious that we have

$$x_s(u_1, \ldots, u_m, \psi_1(u_1), \ldots, \psi_n(u_1), 0) \equiv \varphi_s(u_1, \ldots, u_m) \quad (2.6)$$

In [6] it is shown that if this solution is to be periodic with periods $\omega_2, \ldots, \omega_m$ in the variable $u_2, \ldots, u_m$, then it is necessary and sufficient that the following conditions be fulfilled

$$\begin{aligned} &\gamma_s(u_1, \eta_1(u_1, \alpha), \ldots, \eta_n(u_1, \alpha), \alpha) = \qquad (2.7)\\ &= x_s\left(u_1, \frac{\omega_2}{\omega_1}u_1 + \omega_2, \ldots, \frac{\omega_m}{\omega_1}u_1 + \omega_m, \eta_1, \ldots \eta_n, \alpha\right) - \\ &\qquad - x_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \eta_1, \ldots, \eta_n, \alpha\right) = \\ &= x_s\left(u_1, \frac{\omega_2}{\omega_1}u_1 + \omega_2, \ldots, \frac{\omega_m}{\omega_1}u_1 + \omega_m, \eta_1, \ldots, \eta_n, \alpha\right) - \eta_s(u_1, \alpha) = 0 \end{aligned}$$

Because of (2.6), conditions (2.7) are satisfied if $\alpha = 0$, $\eta_s(u_1, 0) = \psi_s(u_1)$ since the generating equation will be periodic.

Therefore, if condition

$$\left\{\frac{\partial(\gamma_1, \ldots, \gamma_n)}{\partial(\eta_1, \ldots, \eta_n)}\right\} \neq 0 \quad \text{if } \alpha = 0, \quad \eta_s = \psi_s(u_1) \quad (2.8)$$

is fulfilled, then, if $\alpha$ is small enough, equations (2.7) will have one and only one solution $\eta_s(u_1, \alpha)$ for which $\eta_s(u_1, 0) = \psi_s(u_1)$.

Substituting this solution into the function $x_s$, we obtain a solution of (2.2) which is periodic in $u_2, \ldots, u_m$ and reduces to the generating one for $\alpha = 0$. Let us find the condition under which the solution will be periodic in $u_1$. Suppose that

$$\varphi_s\left(\frac{\omega_1}{\omega_2} u_2, u_2, \frac{\omega_3}{\omega_2} u_2, \ldots, \frac{\omega_m}{\omega_2} u_2\right) = \theta_s(u_2)$$

where $\theta_s(u_2)$ is a periodic function of period $\omega_2$. Here $\theta_s(u_2) = \psi_s(u_1)$ if $\omega_2 u_1 = \omega_1 u_2$. Let us denote the solution of equation (2.2) by

$$x_s(u_1, \ldots, u_m, \xi_1(u_2, \alpha), \ldots, \xi_n(u_2, \alpha), \alpha)$$

with the initial condition

$$x_s\left(\frac{\omega_1}{\omega_2} u_2, u_2, \frac{\omega_3}{\omega_2} u_2, \ldots, \frac{\omega_m}{\omega_2} u_2, \xi_1(u_2, \alpha), \ldots, \xi_n(u_2, \alpha), \alpha\right) = \xi_s(u_2, \alpha)$$

Here

$$x_s(u_1, \ldots, u_m, \theta_1(u_2), \ldots, \theta_n(u_2), 0) \equiv \varphi_s(u_1, \ldots, u_m)$$

In order that the solution $x_s$ may be periodic of periods $\omega_1, \omega_3, \ldots, \omega_m$ in the variables $u_1, u_3, \ldots, u_m$ it is necessary and sufficient, just as above, that the following conditions be fulfilled

$$\begin{aligned} &\delta_s(u_2, \xi_1(u_2, \alpha), \ldots, \xi_n(u_2, \alpha), \alpha) = \\ &= x_s\left(\frac{\omega_1}{\omega_2} u_2 + \omega_1, u_2, \frac{\omega_3}{\omega_2} u_2 + \omega_3, \ldots, \frac{\omega_m}{\omega_2} u_2 + \omega_m, \xi_1, \ldots, \xi_n, \alpha\right) - \\ &\qquad - \xi_s(u_2, \alpha) = 0 \end{aligned} \tag{2.9}$$

Conditions (2.9) will be satisfied for $\alpha = 0$, $\xi_s(u_2, 0) = \theta_s(u_2)$. Therefore, if

$$\left\{\frac{\partial(\delta_1, \ldots, \delta_n)}{\partial(\xi_1, \ldots, \xi_n)}\right\} \neq 0 \quad \text{for } \alpha = 0, \quad \xi_s = \theta_s(u_2) \tag{2.10}$$

then equation (2.9) will have, for small enough $\alpha$, one and only one solution $\xi_s(u_2, \alpha)$ for which $\xi(u_2, 0) = \theta_s(u_2)$. Substituting this solution into the function $x_s$, we obtain a solution of equation (2.2) which is periodic in $u_1, u_3, \ldots, u_m$, and which for $\alpha = 0$ reduces to the generating solution.

But the solutions $x_s(u_1, \ldots, u_m, \eta_1, \ldots, \eta_n, \alpha)$ and $x_s(u_1, \ldots, u_m, \xi_1, \ldots, \xi_n, \alpha)$ of equation (2.2) coincide when $u_1/\omega_1 = u_2/\omega_2 = \ldots = u_m/\omega_m$, and, because of the uniqueness of the solution, they coincide also for other values of $u_1, \ldots, u_m$. Thus, if conditions (2.8) and (2.10) are satisfied simultaneously, the solution $x_s(u_1, \ldots, u_m, \eta_1, \ldots, \eta_n, \alpha)$ of equation (2.2) will be periodic in all its variables, and the functions $\eta_s(u_1, \alpha)$, which are periodic in $u_1$ if $\alpha = 0$, will be periodic also for small enough $\alpha$, i.e. the following theorem is true.

*Theorem 2.1.* If conditions (2.8) and (2.10) are fulfilled, then equation (2.2), for small enough $\alpha$, will have one and only one periodic solution which becomes the generating solution when $\alpha = 0$.

Along the diagonal $u_k = t$, the functions (2.4), i.e. $\varphi_s(t, \ldots, t)$, represent the solution of the equation

$$dx / dt = f(t, x) \tag{2.11}$$

This leads to the next result.

*Theorem 2.2.* If equation (2.11) has a quasi-periodic solution $x_s = \varphi_s(t, \ldots, t)$ and if hereby the determinants

$$\left\{\frac{\partial(\gamma_1, \ldots, \gamma_n)}{\partial(\eta_1, \ldots, \eta_n)}\right\}_{\alpha=0,\ \eta_s=\psi_s(t)}, \qquad \left\{\frac{\partial(\delta_1, \ldots, \delta_n)}{\partial(\xi_1, \ldots, \xi_n)}\right\}_{\alpha=0,\ \xi_s=\theta_s(t)}$$

are different from zero, then equation (2.1) has a quasi-periodic solution which becomes the generating one when $\alpha = 0$.

2. Let us assume that the generating equation (2.3) has a family of periodic solutions

$$\varphi_s(u_1, \ldots, u_m, h(u_2 - u_1, \ldots, u_m - u_1)) \qquad (s = 1, \ldots, n) \tag{2.12}$$

which depend on an arbitrary periodic function $h$ of periods $\omega_k - \omega_1$, and that the considered generating solution belongs to this family and corresponds to the function $h = h^*$. Let us denote, as above, by $x_s(u_1, \ldots, u_m, \eta_1(u_1, \alpha), \ldots, \eta_n(u_1, \alpha), \alpha)$ the solution of equation (2.2) with the initial conditions (2.5). Then, just as above, the conditions for periodicity of this solution in $u_2, \ldots, u_m$ will have the form (2.7). These conditions will be satisfied for the generating solution, that is, for

$$\alpha = 0, \qquad \eta_s(u_1, 0) = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu^*\right)$$
$$\left(\mu^* = h^*\left[\left(\frac{\omega_2}{\omega_1} - 1\right)u_1, \ldots, \left(\frac{\omega_m}{\omega_1} - 1\right)u_1\right]\right)$$

But in contradistinction to the preceding case, the conditions (2.7) will be satisfied also when

$$\alpha = 0, \qquad \eta_s(u_1, 0) = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu\right)$$
$$\left(\mu = h\left[\left(\frac{\omega_2}{\omega_1} - 1\right)u_1, \ldots, \left(\frac{\omega_m}{\omega_1} - 1\right)u_1\right]\right)$$

since all solutions (2.12) of the generating equation (2.3) are periodic.

In other words, equations (2.7) have, for $\alpha = 0$, not only one solution, as earlier, but a family of solutions

$$\eta_s(u_1, 0) = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu\right)$$

which depends on an arbitrary function $\mu(u_1)$.

Because of this, the functional determinant

$$\left\{\frac{\partial(\gamma_1, \ldots, \gamma_n)}{\partial(\eta_1, \ldots, \eta_n)}\right\} \quad \text{for } \alpha = 0, \ \eta_s = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu^*\right)$$

will necessarily have to be zero.

Therefore, we shall call the periodic solution (2.4) of equation (2.3) an isolated solution if the determinant (2.8) is different from zero.

The corresponding quasi-periodic solution $\varphi_s(t, \ldots, t)$ of equation (2.11) will in this case also be called an isolated solution.

Thus Theorem 2.2 will take on the following formulation.

For every isolated generating quasi-periodic solution, system (2.1) for sufficiently small $\alpha$ admits one and only one quasi-periodic solution which reduces to the generating one when $\alpha = 0$.

Therefore, in the case of an isolated generating solution, there exists a complete correspondence between systems (2.1) and the simplified system (2.11).

In order to clear up the question on the existence of a periodic solution of equation (2.2) in the case of a family of periodic solutions (2.12), we exclude from equation (2.7) some $(n - 1)$ terms of the quantities $\eta_1, \ldots, \eta_n$. This can always be done when at least one of the minors of order $n - 1$ of the determinant (2.8) is different from zero, which we shall assume to be the case.

Suppose that for the sake of definiteness

$$\left\{\frac{\partial(\gamma_1, \ldots, \gamma_{n-1})}{\partial(\eta_1, \ldots, \eta_{n-1})}\right\} \neq 0 \quad \text{if } \alpha = 0, \ \eta_s = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu^*\right)$$

Under this conditions the first $n - 1$ equations of (2.7) have a solution for $\eta_1, \ldots, \eta_{n-1}$, in which these quantities are functions of $u_1$, $\alpha$ and $\eta_n$, which become

$$\varphi_1\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu^*\right), \ldots, \varphi_{n-1}\left(u_1, \frac{\omega_2}{\omega_1}u_1, \ldots, \frac{\omega_m}{\omega_1}u_1, \mu^*\right)$$

if

$$\alpha = 0, \qquad \eta_n = \varphi_n\left(u_1, \frac{\omega_2}{\omega_1} u_1, \ldots, \frac{\omega_m}{\omega_1} u_1, \mu^*\right)$$

Substituting these quantities into the last one of equations (2.7) we obtain for $\eta_n(u_1, \alpha)$ one equation which can be expressed in the form

$$\Phi = M(\eta_n) + \alpha N(\eta_n, \alpha) = 0 \tag{2.13}$$

Since the system (2.7) has for $\alpha = 0$ the solution

$$\eta_s(u_1, 0) = \varphi_s\left(u_1, \frac{\omega_2}{\omega_1} u_1, \ldots, \frac{\omega_m}{\omega_1} u_1, \mu\right) \qquad (s = 1, \ldots, n)$$

which depends on the arbitrary function $\mu(u_1)$, equation (2.13) must have, under these conditions, one solution $\eta_n$ which depends on an arbitrary function $\mu(u_1)$. This is possible only if the function $M$ vanishes identically. Thus equation (2.13) (after division by $\alpha$) takes on the form

$$N(\eta_n, \alpha) = 0 \tag{2.14}$$

For the existence of a periodic solution it is necessary and sufficient that equation (2.14) have a solution for $\eta_n$ which has to reduce to

$$\varphi_n\left(u_1, \frac{\omega_2}{\omega_1} u_1, \ldots, \frac{\omega_m}{\omega_1} u_1, \mu^*\right) \text{ for } \alpha = 0$$

For this it is necessary, first of all, that the following relation holds

$$P(\mu^*) = N\left(\varphi_n\left(u_1, \frac{\omega_2}{\omega_1} u_1, \ldots, \frac{\omega_m}{\omega_1} u_1, \mu^*\right), 0\right) = 0 \tag{2.15}$$

Thus we have obtained a necessary condition which the function $\mu^*(u_1)$ of the generating solution must satisfy in order that there may correspond to it a periodic solution of the complete system (2.2). If in addition to condition (2.15) we have also the condition

$$\left\{\frac{\partial N}{\partial \eta_n}\right\} \neq 0 \quad \text{for } \alpha = 0, \qquad \eta_n = \varphi_n\left(u_1, \frac{\omega_2}{\omega_1} u_1, \ldots, \frac{\omega_m}{\omega_1} u_1, \mu^*\right)$$

then equation (2.15) will have a unique solution for $\eta_n$ of the required type, and then, as was indicated above, there will exist a periodic solution of equation (2.2).

We note that if the function $\mu^*(u_1)$ is known, then the functions $h^*(u_2 - u_1, \ldots, u_m - u_1)$ can be determined in a unique way since $Dh^* = 0$.

Along the diagonal $u_k = t$, the functions (2.12), that is,

$$\varphi_s(t, \ldots, t, l) \tag{2.16}$$

represent a family of quasi-periodic solutions of equation (2.11) which depend on an arbitrary parameter $l$. Thus we have obtained the next theorem.

*Theorem 2.3.* In the infinite set of generating quasi-periodic solutions in the family (2.16) of equation (2.11) there are only certain ones which can actually correspond to quasi-periodic solutions of the original system (2.1); namely, only those solutions for which the parameter $l^*$ takes on certain definite numerical values.

The method used in the investigation of equation (2.1) can be used also for the investigation of the autonomous equation

$$\frac{dx}{dt} = f(x) + \alpha F(x, \alpha)$$

The application of this method shows that the theory of a small parameter of Poincaré for periodic solutions [7] can be used, without essential modifications, for the study of quasi-periodic cases.

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